[CONTRIBUTION FROM THE JOHN HARRISON LABORATORY OF THE UNIVERSITY OF PENNSYLVANIA]

A Method for Calculating the Effect of Residual Inductances in High Frequency Capacitance Measurements

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The growing interest in the dielectric polarization of pure liquids and liquid mixtures demands improvement in the accuracy of the measurements of dielectric constant. The status of the methods now employed is revealed in the lack of agreement in the reported values of the dielectric constants of even the best standard liquids such as benzene and carbon tetrachloride.

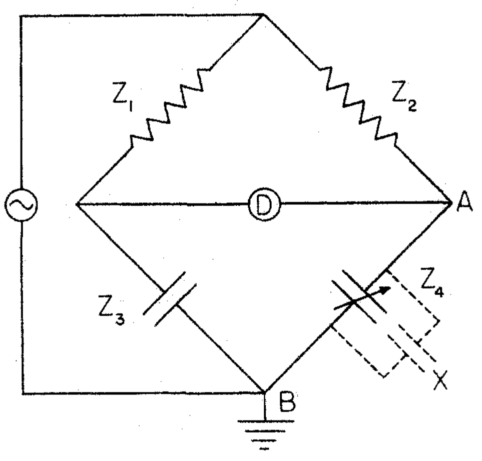


Fig. 1.

The advantages of high frequency measurements of capacitance have led to their widespread use in the determination of dielectric constants. Unfortunately, many of these measurements have been made without complete consideration of the networks employed. The neglect of residual reactances, especially of inductances, would account for many errors in the calculations made from such measurements.

Some methods[1] have been produced for minimizing the effects of inductive residuals and for calculating these effects in part. That these methods could be improved was shown by the work of Field and Sinclair[2] in determining the residuals in a variable air condenser of the precision type usually used as a standard in measurements of capacitance.

In the following treatment of the residual reactances in a capacitance bridge network there is presented a method which should be of definite assistance in improving the accuracy of measurements of dielectric constants of liquids of low conductance.

Theory of the Bridge Network

A simple capacitance bridge of the form shown in Fig. 1 is of the type commonly used to determine capacitance where the dielectric and ohmic losses of the condensers involved are negligible. With adequate shielding, true balance is obtained when the impedances are adjusted so that $Z_1Z_4 = Z_2Z_3$. The ratio arm impedances Z_1 and Z_2 are usually equal resistance units of matched characteristics. Their optimum magnitude is inversely proportional to the frequency used, so that at high frequencies their required value is low enough to render easy the matching and true balance is assured when the impedance, Z_3, of the balancing arm is equal to that, Z_4, of the measuring arm. If the condensers are loss-free, true balance can then be obtained by capacitance adjustment alone. For greatest flexibility, variable air condensers should be placed in these arms.

In order to perform an accurate analysis of the network at high frequencies, we must consider the fact that residual reactances are present not only in the leads but also in each condenser even though the condensers closely approach ideality as circuit elements. To aid the analysis we may lump the residuals. Thus, the capacitance, C_s, of the condenser in the balancing arm can be pictured in series with the total equivalent inductance, L_a, of its arm and shunted by the total parasitic capacitance, C_a, of that arm. The effective capacitance, $\hat{C}$, of that arm at any frequency, f, will be given by

$$\hat{C}_s = C_a + [C_s/(1 - \omega^2 L_a C_s)] \text{ (where } \omega = \text{ angular velocity } = 2\pi f)$$

Consider the measuring arm, AB, arranged to measure the unknown capacitance of the loss-free condenser X. This measurement may be made most accurately by the parallel substitution method. A variable air condenser is placed permanently in this arm throughout the measurements. The leads to the unknown are attached

(1) See, for example: Davies, *Phil. Mag.*, **20**, 75 (1935); Lattey and Gatty, *ibid.*, **7**, 985 (1929); Watson, *Proc. Roy. Soc.* (London), **A117**, 43 (1927).

(2) Field and Sinclair, *Proc. Inst. Radio Engrs.*, **24**, 255 (1936).

to the terminals of this air condenser and should be rigid enough to hold their positions in the absence of the unknown. With the balancing arm fixed, the balance readings, C and C', of the air condenser in the absence and in the presence of the unknown in the circuit allow calculation of the capacitance, C_x, of the unknown.

The calculation is commonly made with neglect of the effect of the inductive residuals. Considering the capacitance of the variable condenser to be shunted by the capacitance, C_b, of its leads to the bridge terminals and by that, C_c, of the leads to the unknown, we would have, neglecting the inductances present

$$C_a + C_s = C + C_b + C_c$$

and

$$C_a + C_s = C' + C_b + C_c + C_x$$

with the result that, on equating the measuring arm capacitances, the unknown would be calculated by the equation

$$C_x = C - C' \qquad (1)$$

The measuring arm connections are more accurately represented in Fig. 2. The capacitances are shown as described before, but the network now shows the inductances, L_b and L_c, of the leads and the inductances, L and L_x, contained within the variable air condenser and the unknown, respectively.

As before, we may equate the input capacitances for the measuring arm when the balancing arm is fixed. Obeying Fig. 2, we have

$$C_b + \frac{C}{1 - \omega^2(L_b + L)C} + \frac{C_c}{1 - \omega^2(L_b + L_c)C_c} = C_b + \frac{C'}{1 - \omega^2(L_b + L)C'} + \frac{C_c}{1 - \omega^2(L_b + L_c)C_c} + \frac{C_x}{1 - \omega^2(L_b + L_c + L_x)C_x}$$

Therefore

$$\frac{C}{1 - \omega^2(L_b + L)C} = \frac{C'}{1 - \omega^2(L_b + L)C'} + \frac{C_x}{1 - \omega^2(L_b + L_c + L_x)C_x} \qquad (2)$$

It is apparent that considerable error would result in the use of equation (1) at high frequencies. This error would be hidden where a single frequency is used and the balancing arm is fixed, but would be revealed on changing either the frequency of the measurement or the setting of the balance arm.

The Method

Evaluation of C_x by equation (2) would be tedious and would require accurate knowledge of the inductances involved. These difficulties may be removed by rearranging that equation with the aid of several allowable approximations. Since

$$\frac{C}{1 - \omega^2(L_b + L)C} - \frac{C'}{1 - \omega^2(L_b + L)C'} \cong \frac{C - C'}{1 - \omega^2(L_b + L)(C + C')}$$

we have

$$\frac{C - C'}{1 - \omega^2(L_b + L)(C + C')} = \frac{C_x}{1 - \omega^2(L_b + L_c + L_x)C_x} \qquad (3)$$

Equation (3) may be used in several ways to obtain C_x. We may rewrite this equation

$$C - C' = C_x\left[\frac{1 - \omega^2(L_b + L)(C + C')}{1 - \omega^2(L_b + L_c + L_x)C_x}\right] \cong C_x + \omega^2[(L_b + L_c + L_x)C_x^2 - (L_b + L)(C + C')C_x]$$

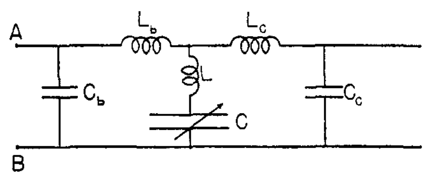
so that

$$C - C' = C_x + \omega^2\left[L_b + L_c + L_x - (L_b + L)\frac{C + C'}{C_x}\right]C_x^2 \qquad (4)$$

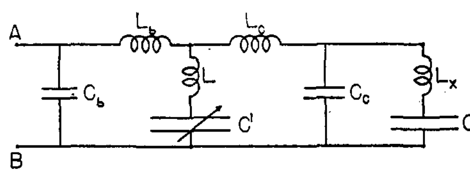
The term in square brackets may be called the equivalent inductance of the measuring arm and may be symbolized L_y. Writing $\omega = 2\pi f$, we obtain

$$C - C' = C_x + 4\pi^2 L_y C_x^2 f^2 \qquad (5)$$

This equation shows that a plot of $C - C'$ against f^2 should, if L_y remains constant, give a straight line whose intercept is C_x and whose slope is $4\pi^2 L_y C_x^2$.



Unknown disconnected.



Unknown connected.

Fig. 2.

If equation (3) were rewritten

$$C_x = (C - C')\frac{1 - \omega^2(L_b + L_c + L_x)C_x}{1 - \omega^2(L_b + L)(C + C')} \cong (C - C')[1 - \omega^2(L_b + L_c + L_x)C_x + \omega^2(L_b + L)(C + C')] \qquad (6)$$

then, on defining L_y as before, we obtain

$$C - C' = C_x/(1 - 4\pi^2 L_y C_x f^2) \qquad (7)$$

This equation shows that a plot of $1/(C-C')$ against f^2 should give a straight line whose intercept is $1/C_x$ and whose slope is $-4\pi^2 L_y$.

Equation (7) will generally be more accurate than equation (5) due to the fact that the approximation made in obtaining equation (6) will generally be more allowable than that made in obtaining equation (4). In the former case the approximation neglects the term

$$\omega^4(L_b + L)^2 (C + C')^2 - \omega^4(L_b + L_c + L_x)(L_b + L) (C + C')C_x$$

while that of the latter neglects

$$\omega^4(L_b + L_c + L_x)^2C_x^2 - \omega^4(L_b + L_c + L_x)(L_b + L) (C + C')C_x$$

Outside of this the equations have the same basis. These approximations show that, unless C_x is very small, equation (7) will be preferable to equation (5) since it will generally be true that $L_x \gg L$. The comparatively large self-contained inductance of the ordinary types of measuring cell would lead to the use of equation (7) in dielectric constant measurements.

This method of frequency variation is conditioned largely upon the constancy of L_y. In the first place, it is not to be expected that the individual inductances L, L_b, L_c and L_x will remain constant at all frequencies. It was found, however, that their variations are negligible over a region of frequencies wide enough to provide accurate extrapolation. It is also true that the term $(C + C')/C_x$ in L_y will vary with change in frequency due to the unavoidable changes in C'. To minimize this effect upon L_y it is best to keep C large and constant. In doing this we must consider the balancing arm, the variable air condenser of which may not be of the precision type. The effective capacitance of this arm will, in general, vary with frequency at a rate different from that of the effective capacitance of the measuring arm, due to lack of symmetry in the two arms. The balancing arm must be set near its maximum to make C large and its residual reactances, due to L_a and C_a, should not be greatly different from those in the measuring arm in the absence of the unknown. In this case, C can be kept constant by small changes in the setting of the balancing arm as the frequency is changed. Reproducibility of the settings of the balancing arm will be maintained at a maximum in this way. Actually, it was found that it is an easy matter to arrange the leads in the measuring arm to obtain symmetry sufficient to keep C constant for a permanent setting of the balancing arm. In this case, the measurements are facilitated and errors due to lack of reproducibility of C_s disappear.

Minimizing of residual reactances in both arms in the absence of the unknown not only aids in producing symmetry but also lowers the variance in L_y by reducing L_b in the term

$$(L_b + L)\,\frac{C + C'}{C_x}$$

As pointed out above, the error in equation (1) would appear at a single frequency if an attempt were made to check the value of C_x by varying the balancing arm to obtain several readings of $C - C'$. In this case, variance of $C - C'$ would show the effect of the inductive residuals on the variable condenser only, since the effective capacitance, $\hat{C}_x$, of the unknown would remain constant at a fixed frequency as shown by equation (3)

$$\frac{C - C'}{1 - \omega^2(L + L_b)(C + C')} = \frac{C_x}{1 - \omega^2(L_b + L_c + L_x)C_x} \equiv \hat{C}_x$$

According to this equation, a plot of $C - C'$ against $C + C'$ would give a straight line whose intercept is $\hat{C}_x$ and whose slope is $-\omega^2(L + L_b)\hat{C}_x$. This method was developed by Field and Sinclair,[2] who used it to evaluate the residual inductance in the variable air condenser. It is apparent that this method does not give the true value of C_x but would give an accurate value of $\hat{C}_x$ at the fixed frequency. Values of $\hat{C}_x$ obtained in this way could be used as follows in the determination of dielectric constants.

The total capacitance of the measuring cell is made up of a fixed part due to its self-contained leads and stray capacitances and a replaceable part which depends upon the dielectric constant of the substance filling the cell. If we group the fixed capacitances in the term C_0 and symbolize the replaceable capacitance C_r, we see that $C_x = \epsilon C_r + C_0$, where ϵ is the dielectric constant of the substance filling the cell. Thus, at any frequency

$$\hat{C}_x = \frac{\epsilon C_r + C_0}{1 - \omega^2(L_b + L_c + L_x)(\epsilon C_r + C_0)}$$

so that if the cell is filled in turn with three different substances of known dielectric constants, the resultant values of $\hat{C}_x$ at the same frequency and with the same leads will present accurate knowledge of C_r, C_0 and $L_b + L_c + L_x$ for the determination of the dielectric constant of any other substance.

While this method is based on an equation obtained by fewer approximations than those of the frequency variation method, calculations of C_x based upon it would be more tedious. Furthermore, the variation of the balancing arm might entail the introduction of considerable error on many capacitance bridges. For this reason the method was not tried here, although Field and Sinclair, using a Schering bridge, have found it very satisfactory.

Both the constant capacitance method and the constant frequency method could be applied to the tuned circuit and beat frequency methods of capacitance measurement, where inductive residuals are important.

Experimental

The bridge that was used met the requirements of the foregoing analysis. It was a General Radio Company 516-A instrument.[3] Its shielding was quite adequate and its ratio arm elements were 100 ohm resistances carefully matched and symmetrically placed. The resistance decade in the balancing arm was placed in series with the condenser of that arm which was set at its maximum, about 1000 $\mu\mu$f. In the measurements quoted, only a small part of the one ohm slide wire in the resistance decade was needed.

The measuring arm of the bridge was constructed as follows. The precision condenser was fixed firmly in place, being connected to the bridge terminals by rigid leads of heavy brass wire. These leads were arranged to present a minimum of inductance and capacitance. A few trials found an arrangement such that the precision condenser setting in the absence of the unknown remained constant over the band of frequencies used in the measurements. The measuring cell, of the type designed by Smyth and Morgan,[4] was held in a glass tray in an oil filled thermostat kept constant within 0.05°. The leads from the terminals of the precision condenser to the mercury cup terminals of this cell were of heavy brass and were parallel throughout their length which was 32 cm. each. The open-circuit capacitance, C_0, of these leads was 5.0 $\mu\mu$f and, due to its small size, independent of the frequency. The value of $L_0 + L_x$ was about 1.0 microhenry and $L + L_b$ was about 0.1 microhenry. The fixed capacitance of the cell was about 8 $\mu\mu$f and the replaceable capacitance was about 177 $\mu\mu$f. All low potential terminals in the network were connected to a common grounding point.

The power source was a General Radio Company 484-A modulated oscillator and was used at frequencies from 0.6 to 1.5 megacycles. Its low power and conveniently variable frequency made it ideal for the measurements. The null detector was a superheterodyne receiver adequately shielded against undesirable pickup. Its sensitivity permitted hearing the null point on its loud speaker. The null point was extremely sharp and true balance was obtained in all readings which were as accurate as the precision condenser, a General Radio 222-M instrument, would permit.

(3) Burke, *General Radio Experimenter*, **7**, 1 (July, 1932).

(4) Smyth and Morgan, THIS JOURNAL, **50**, 1547 (1928).

The measuring cell was filled with air or liquids of low dielectric constant and negligible conductance. These liquids were benzene, carbon tetrachloride, germanium tetrachloride and mixtures of these. The preparation of these liquids and the method of filling the measuring cell were as has already been described.[5]

Results and Discussion

Table I lists typical results of the measurements of the unknown capacitance of the measuring cell. The variation with frequency is very small for the air filled cell but quite appreciable for the cases of higher capacitance. The data for the carbon tetrachloride and germanium tetrachloride fillings are plotted, as B and C, respectively, in Fig. 3 according to equation (5) and in Fig. 4 according

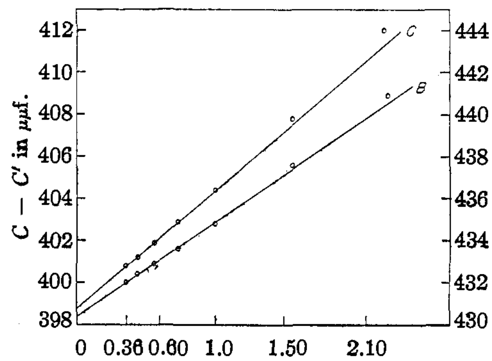


Fig. 3.

to equation (7). These figures show that equation (7) fits the data slightly better than does equation (5) since in Fig. 3 the higher frequency values tend to curve off the line more than those in Fig. 4.

TABLE I

THE FREQUENCY VARIATION OF THE APPARENT CAPACITANCE OF THE UNKNOWN

Frequency (megacycles)	$C - C'$ (in $\mu\mu$f) A	B	C
0.600	184.8	400.0	432.8
.666	184.8	400.4	433.2
.750	184.9	400.9	433.9
.857	184.9	401.6	434.9
1.000	185.0	402.8	436.4
1.250	185.3	405.6	439.8
1.500	185.8	408.9	444.0

A: Cell filled with air at 25°.
B: Cell filled with carbon tetrachloride at 35°.
C: Cell filled with germanium tetrachloride at 40°.

The values of C_x and L_y obtained from the intercepts and slopes in Fig. 4 are 398.30 $\mu\mu$f and 0.73 μh for the carbon tetrachloride filling and 430.-70 $\mu\mu$f and 0.78 μh for the geramanium tetrachlo-

(5) Miller, *ibid.*, **56**, 2360 (1934).

ride filling. These agree fairly well with the corresponding values, 398.36 $\mu\mu$f and 0.72 μh, 430.74 $\mu\mu$f and 0.77 μh, obtained from Fig. 3, showing that either of these methods of extrapolation is permissible.

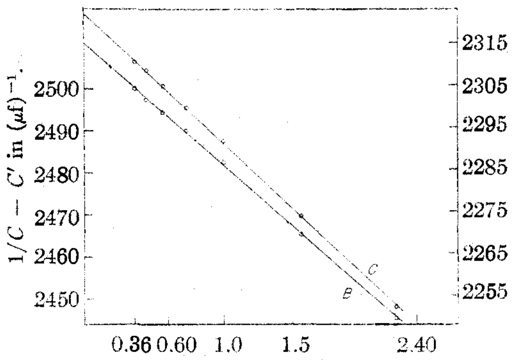


Fig. 4.

The data for the air-filled cell when plotted give similar results placing C_x at 184.66 $\mu\mu$f and L_y at 0.23 μh. Many other measurements were made and all confirmed the method which permitted estimating C_x within 0.05 $\mu\mu$f. This accuracy is well within the experimental error of the precision condenser readings and could be improved by improvement of the precision condenser calibration. The values of L_y were always consistent with the values of C_x.

The error in the value of C_r will introduce the largest error in the calculation of dielectric constant by the equation $\epsilon = (C_x - C_0)/C_r$. Since the error in C_r will be $\pm$ 0.05 $\mu\mu$f, the percentage error using a measuring cell whose replaceable capacitance is 177 $\mu\mu$f would be considerably less than 0.05%. This would permit estimation of dielectric constants with a similar accuracy, *i. e.*, within better than 1 part in 2000.

It is true that comparatively small errors will be caused by the neglect of residual inductances in the calculation of dielectric constants of almost the same value as that of the standard used to calibrate the cell. With increasing difference between the dielectric constant of the measured substance and that of the standard these errors increase rapidly. Thus, using a standard of dielectric constant 2.273, the error caused by neglect of residual inductances in the calculation of the dielectric constant of a liquid whose true value was 2.426 was found to be 0.17% at 1 megacycle and 0.37% at 1.5 megacycles. That these errors were not experimental was shown by the excellence with which the data fitted equations (5) and (7).

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Summary

1. Analysis is made of the effects of residual inductances in the measurement of a capacitance at high frequencies.

2. A method is proposed for calculating the true value of the capacitance of a cell used in measuring the dielectric constant of liquids of low conductance. This method is based on parallel substitution measurements made at varying frequency and constant total capacitance.

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